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Thermal analysis of a longitudinal fin with variable thermal properties by recursive formulation

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Abstract

The present study supplies a new approach to calculate thermal performance of a singular fin with variable thermal properties. With discrete model, the singular fin can be divided into many sections. Then, each section can be combined together to obtain the whole solution of the fin by recursive numerical formulation. The recursive formulas for both conditions with and without heat transfer on fin tip are derived in the present study. Finally, several examples including composite and boiling mode of a fin have been successfully simulated to demonstrate the validity of the present approach.

Keywords: Longitudinal fin; Variable heat transfer coefficient; Variable thermal conductivity

1. Introduction

The heat sink assembly is a commonly used and powerful apparatus for heat removal in today's thermal engineering. This apparatus is mainly designed to remove heat effectively from the hot equipment to environment. In all types of heat sink assembly, fins play a very important role. The analysis of fin is of the interest that the extended surface can promote heat transfer. During the process for designing fins, material weight and manufacturing cost are the first to be concerned.

In 1972, Kern and Kraus [1] presented a series of study on extended surface. And Kraus [2] presented a fin literature consists of what scholars had done for the past 65 years. Generally speaking, the thermal behavior by using fin dissipation in real world is complicated because of its variable thermal properties. For example, heat transfer coefficient on fin surface and thermal conductivity of fin material are usually temperature-dependent and location-dependent. And fin material is neither homogeneous nor isotropic. Also, fin profile is varied with different demands and hence fin cross-section area may be uniform or step changed. Gardner [3] is the first one to solve this problem with constant heat transfer coefficient and thermal conductivity. By using the simplified one-dimensional fin model, optimum solution can be determined easily and that is widely used in recent studies and industrial practice. In the thermal analysis of a fin behavior, one-dimensional approach is usually limited for some specified cases. It is found that one-dimensional model of fin performance is valid as transverse Biot number is less than unity by

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Nomenclature

a	normal fin cross-section area, m^2
c	dimensionless parameter for exponential increase in heat transfer coefficient, defined in Eq. (37)
h	heat transfer coefficient, W/m^2 K
k	thermal conductivity, W/m K
k_0	thermal conductivity at T_0, W/m K
l	length of fin, m
m	fin parameter, defined in Eq. (1)
M	dimensionless parameter, defined in Eq. (3)
N	dimensionless parameter, defined in Eq. (30)
p	perimeter of fin, m
q	heat transfer rate of fin, W
R	thermal resistance of singular fin, K/W
T	temperature of fin surface, K
V	fin volume, m^3
W	fin parameter, defined in Eq. (5)
x	distance from fin base, m

Greek symbols

γ	fin width, m
δ	fin thickness, m
ε	slope of temperature-dependent thermal conductivity curve
θ	temperature difference between fin surface and environment, K
μ	thermal transmission ratio, defined as the reciprocal of thermal resistance, W/K
ρ	dimensionless parameter for heat transfer coefficient, defined in Eq. (39)

Superscripts

σ	dimensionless parameter for heat transfer coefficient, defined in Eq. (39)
m	power of temperature difference
n	power of characteristic length of fin

Subscripts

0	base
av.	average
b	base temperature difference in excess of saturated temperature
Copper	material of copper
c	characteristic length
Dural.	material of Duralumin
d	dryout location
e	fin tip
f	film boiling location
i	an index for discrete model, $i = 1,2,3, \ldots$
j	an index refers to the type of boiling, defined in Eq. (39)
l	fin length
n	number of sections for discrete model
α	dimensionless parameter for exponential increase in heat transfer coefficient, defined in Eq. (37)
β	dimensionless parameter for exponential increase in heat transfer coefficient, defined in Eq. (36)
∞	surrounding

Irey [4]. To expand this valid range, an improved one-dimensional solution was proposed by Aparecido and Cotta [5] who modified those expressions as simple as classical ones but accuracy is significantly improved.

Considering a fin with variable thermal conductivity, Aziz and Huq [6] presented a perturbation solution and check the accuracy of the solution with numerical method. Concerning arbitrary variable heat transfer coefficient on fin surface, Ma et al. [7] applied Fourier series approach to investigate a two-dimensional rectangular fin. Laor and Kalman [8] considered a power-law type, temperature-dependent heat transfer coefficient, the solution could be solved numerically by a well recognized technique. With a prescribed heat flux at fin end in one-dimensional model, Liaw and Yeh [9] calculated temperature distribution and heat transfer rate by solving hypergeometric function. The heat transfer coefficient is assumed to vary with a power-law-type formula. Heat transfer from the fin tip was also concerned. Results from Liaw and Yeh [9] are also compared with experimental data. In 1996, Yeh [10] presented a modified one-dimensional solution for fin optimization.

The purpose of this study is to focus on the longitudinal fin where fin parameters such as heat transfer coefficient and thermal conductivity can be temperature-dependent or position-dependent functions. In this model, the whole fin is separated into many sections where each section can have its local heat transfer coefficient and local thermal conductivity, h_1 and k_1, h_2 and k_2, h_3 and k_3, etc. Although all equations quoted from sections are based on constant thermal properties, each section can be combined together to obtain the whole solution of the fin by recursive numerical formulation. And the close form solution is easily and straightly yielded by numerical iterative method. By substituting the variable functions into discrete model, the thermal behavior of fin including temperature distribution and heat transfer rate can be obtained.

2. Theoretical analysis

A singular fin with its geometry, material properties, and convective condition to the environment is shown in Fig. 1. And one-dimensional steady state solution including three different boundary conditions on fin tip has been summarized by Incropera and DeWitt [11] shown in Table 1. These results will be utilized into the derivation of generalized recursive formula appeared in the following sections.

2.1. Insulated fin tip

(1) One uniform section where $0 \leqq x \leqq l_1$.

Based on Fig. 2(a) as fin has one uniform section, the governing equation is shown as

$$\frac{d^2\theta}{dx^2} - m_1^2\theta = 0, \quad 0 \leqq x \leqq l_1 \tag{1}$$

where $m_1^2 = \frac{h_1 p}{k_1 a}$

And boundary conditions are set as

$$\theta(0) = \theta_0, \quad \text{and} \quad \left.\frac{d\theta}{dx}\right|_{x=l_1} = 0 \tag{2}$$

The temperature distribution is obtained as

$$\theta(x) = \theta_0 \cdot \frac{\cosh[m_1(l_1 - x)]}{\cosh(M_1)} \tag{3}$$

where $M_1 = m_1 \cdot l_1$.

And the temperature difference on fin tip is

$$\theta_1 = \theta_0 \cdot \frac{1}{\cosh(M_1)} \tag{4}$$

The solution of heat transfer rate in one uniform section, l_1, is

$$q_0 = -k_1 a \left.\frac{d\theta}{dx}\right|_{x=0} = \theta_0 W_1 \tanh(M_1) \tag{5}$$

where $W_1 = \sqrt{h_1 k_1 p a}$.

(2) Two uniform sections where $0 \leqq x \leqq (l_1 + l_2)$.

Model where fin is divided into two uniform sections is shown in Fig. 2(b). As already indicated, heat transfer coefficients and thermal conductivities can be specified with different values of h_1, h_2 and k_1, k_2.

(a) For the section of $0 \leqq x \leqq l_1$.

With a prescribed temperature difference, θ_1, at the end of the first section, the governing equation of section l_1 is given as

$$\frac{d^2\theta}{dx^2} - m_1^2\theta = 0, \quad 0 \leqq x \leqq l_1 \tag{6}$$

where $m_1^2 = \frac{h_1 p}{k_1 a}$.

And boundary conditions are set as

$$\theta(0) = \theta_0, \quad \text{and} \quad \theta(l_1) = \theta_1 \tag{7}$$

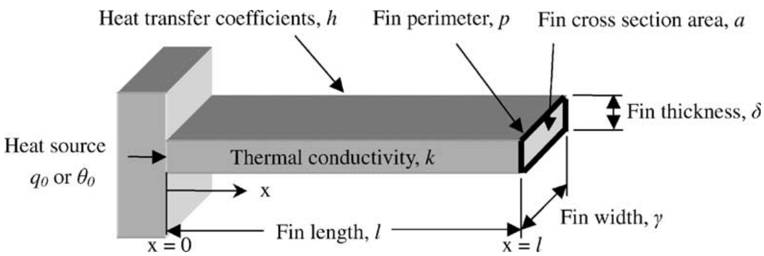

Fig. 1. Models for longitudinal fin structure.

Table 1
Temperature distribution and heat loss for fins for constant cross-sectional area a, perimeter p, thermal conductivity k, and heat transfer coefficient $h_{av.}$

Case	Tip condition, $x = l$	Temperature distribution, $\theta(x)$	Temperature of tip, $\theta(l) = \theta_{tip}$	Fin heat transfer, $q = -ka\frac{d\theta}{dx}\|_{x=0}$
A.	Convective heat transfer $-k\frac{d\theta}{dx}\|_{x=l} = h\theta(l)$	$\theta_0 \cdot \frac{\cosh[m(l-x)] + N\sinh[m(l-x)]}{\cosh(M) + N\sinh(M)}$	$\theta_0 \cdot \frac{1}{\cosh(M) + N\sinh(M)}$	$\theta_0 W\left[\frac{\tanh(M) + N}{1 + N\tanh(M)}\right]$
B.	Adiabatic $\frac{d\theta}{dx}\|_{x=l} = 0$	$\theta_0 \cdot \frac{\cosh[m(l-x)]}{\cosh(M)}$	$\theta_0 \cdot \frac{1}{\cosh(M)}$	$\theta_0 W \tanh(M)$
C.	Prescribed temperature $\theta(l) = \theta_l$	$\{\theta_0 \sinh[m(l-x)] + \theta_l \sinh(mx)\} \cdot \frac{1}{\sinh(M)}$	θ_l	$W_0 \frac{\theta_0 \cosh(M) - \theta_l}{\sinh(M)}$

Where $\theta(0) = T_0 - T_\infty = \theta_0$, $m = \sqrt{\frac{h_{av.}p}{ka}}$, $M = m \cdot l$, $W = \sqrt{h_{av.}pka}$ and $N = \frac{h_e}{mk}$.

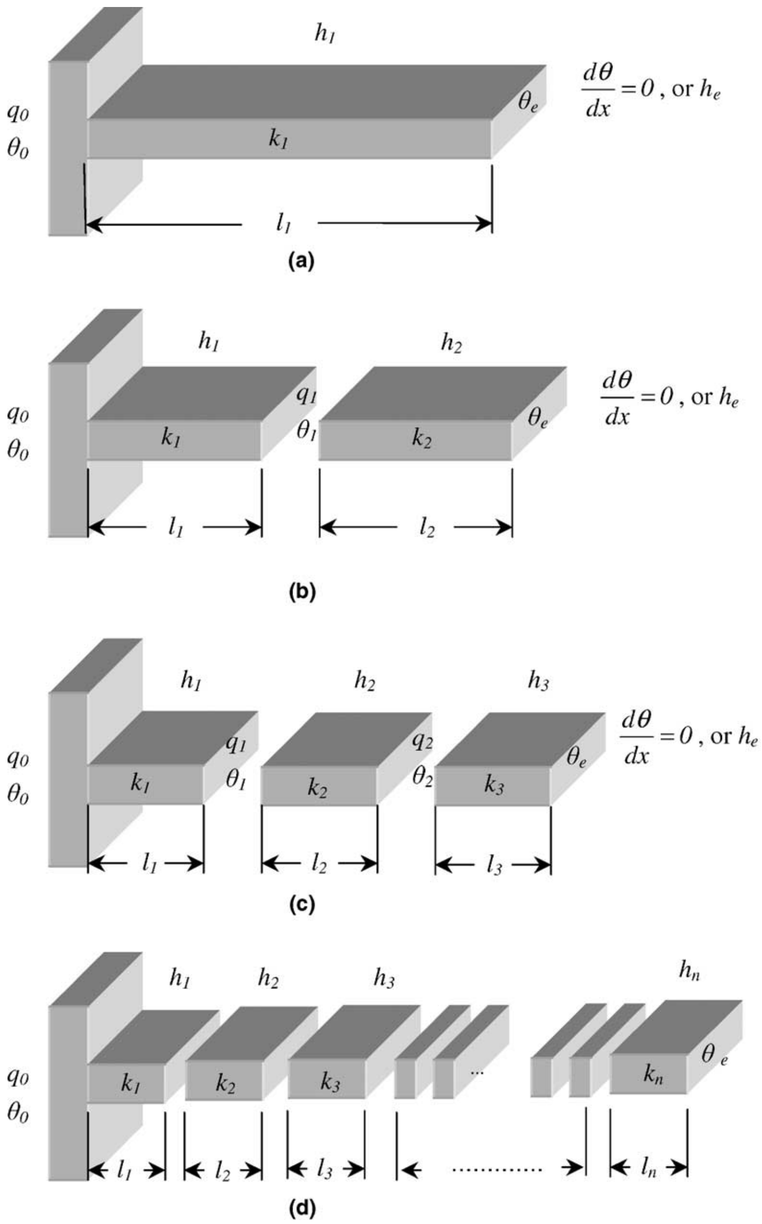

Fig. 2. Discrete models for fin tip with or without heat transfer: (a) one section; (b) two sections; (c) three sections; (d) N sections.

The temperature distribution is obtained as

$$\theta(x) = \frac{\theta_0 \sinh[m_1(l_1 - x)] + \theta_1 \sinh(m_1 x)}{\sinh(M_1)} \tag{8}$$

where $M_1 = m_1 \cdot l_1$.

And both of the heat transfer rates at $x = 0$ and $x = l_1$ are obtained in the form

$$q_0 = W_1 \frac{\theta_0 \cosh(M_1) - \theta_1}{\sinh(M_1)} \tag{9}$$

$$q_1 = W_1 \frac{\theta_0 - \theta_1 \cosh(M_1)}{\sinh(M_1)} \tag{10}$$

where $W_1 = \sqrt{h_1 k_1 p a}$.

And the net heat transfer rate perpendicular to the axis of the first section, l_1, is given as

$$q_0 - q_1 = (\theta_0 + \theta_1) W_1 \tanh\left(\frac{M_1}{2}\right) \tag{11}$$

(b) For the section of $l_1 \leqq x \leqq (l_1 + l_2)$.

Section l_2 of the fin can be determined in the same manner. The governing equation and boundary conditions are expressed as

$$\frac{d^2\theta}{dx^2} - m_2^2\theta = 0, \quad l_1 \leqq x \leqq (l_1 + l_2) \tag{12}$$

where $m_2^2 = \frac{h_2 p}{k_2 a}$.

And boundary conditions are set as

$$\theta(l_1) = \theta_1, \quad \text{and} \quad \left.\frac{d\theta}{dx}\right|_{x=l_1+l_2} = 0 \tag{13}$$

The temperature distribution is obtained as

$$\theta(x) = \theta_1 \cdot \frac{\cosh[m_2(l_1 + l_2 - x)]}{\cosh(M_2)} \tag{14}$$

where $M_2 = m_2 \cdot l_2$.

And temperature on fin tip is

$$\theta_2 = \theta_1 \cdot \frac{1}{\cosh(M_2)} \tag{15}$$

The heat transfer rate in the second section, l_2, is

$$q_1 = \theta_1 W_2 \tanh(M_2) \tag{16}$$

where $W_2 = \sqrt{h_2 k_2 p a}$.

(c) For the section of $0 \leqq x \leqq (l_1 + l_2)$.

Replacing Eq. (16) into Eq. (11), then Eq. (11) can be obtained as

$$q_0 = \theta_0 W_1 \tanh\left(\frac{M_1}{2}\right) + \theta_1\left[W_1 \tanh\left(\frac{M_1}{2}\right) + W_2 \tanh(M_2)\right] \tag{17}$$

And from Eq. (9), θ_1 can be rewritten as

$$\theta_1 = \theta_0 \cosh(M_1) - \frac{q_0 \sinh(M_1)}{W_1} \tag{18}$$

Finally, by substituting Eq. (18) into Eq. (17), the heat transfer rate on fin base, q_0, becomes

$$q_0 = \theta_0 W_1 \frac{\tanh(M_1) + \frac{W_2}{W_1}\tanh(M_2)}{1 + \tanh(M_1)\frac{W_2}{W_1}\tanh(M_2)} \tag{19}$$

By substituting Eq. (18) into Eq. (15), it derives

$$\theta_2 = \theta_0 \frac{\cosh(M_1)}{\cosh(M_2)} - \frac{q_0 \sinh(M_1)}{W_1 \cosh(M_2)} \tag{20}$$

Based on Eq. (19), heat transfer rate from fin surface to surrounding is determined by incorporating the local heat transfer coefficients h_1 and h_2 and conductivities k_1 and k_2 which are included in M_1, M_2, W_1, and W_2. Once one of these two parameters, θ_0 and q_0, is known in advance, the other one can be obtained by Eq. (19). The temperature difference θ_1 can be determined by Eq. (18), and then heat transfer rate q_1 can also be obtained from Eq. (10).

(3) Three uniform sections where $0 \leqq x \leqq (l_1 + l_2 + l_3)$.

The model including three different sections is depicted in Fig. 2(c). Similar to the previous derivation, the solution procedures for these three sections, l_1, l_2 and l_3, are in the same manner.

Here, solutions for three sections, l_1, l_2 and l_3, are obtained, respectively, as

$$q_0 - q_1 = (\theta_0 + \theta_1)W_1 \tanh\left(\frac{M_1}{2}\right) \tag{11}$$

$$q_1 - q_2 = (\theta_1 + \theta_2)W_2 \tanh\left(\frac{M_2}{2}\right) \tag{21}$$

$$q_2 = \theta_2 W_3 \tanh(M_3) \tag{22}$$

Summation of Eqs. (11), (21), and (22) can obtain the heat transfer rate on fin base as :

$$\begin{aligned} q_0 = {} & \theta_0 W_1 \tanh\left(\frac{M_1}{2}\right) \\ & + \theta_1\left[W_1 \tanh\left(\frac{M_1}{2}\right) + W_2 \tanh\left(\frac{M_2}{2}\right)\right] \\ & + \theta_2\left[W_2 \tanh\left(\frac{M_2}{2}\right) + W_3 \tanh(M_3)\right] \end{aligned} \tag{23}$$

As mentioned above, heat transfer rate, q_0, in Eq. (23) has to be expressed in terms of θ_0. Based on Eq. (18), similar solution procedure of temperature from the second section is introduced here as

$$\theta_2 = \theta_1 \cosh(M_2) - \frac{q_1 \sinh(M_2)}{W_2} \tag{24}$$

Substituting Eqs. (10), (18) and (24) into Eq. (23), it can be rearranged as

$$q_0 = \theta_0 W_1 \times \frac{\tanh(M_1) + \frac{W_2}{W_1}\left[\frac{\tanh(M_2) + \frac{W_3}{W_2}\tanh(M_3)}{1 + \tanh(M_2)\frac{W_3}{W_2}\tanh(M_3)}\right]}{1 + \tanh(M_1)\frac{W_2}{W_1}\left[\frac{\tanh(M_2) + \frac{W_3}{W_2}\tanh(M_3)}{1 + \tanh(M_2)\frac{W_3}{W_2}\tanh(M_3)}\right]} \tag{25}$$

(4) N uniform sections where $0 \leqq x \leqq (l_1 + l_2 + l_3 + \cdots + l_n)$.

From the recursive relation among Eqs. (5), (19), and (25), It is noted that the same manner can be extended for more sections which are shown in Fig. 2(d). For example, a fin can be separated into one, two, three, or four sections and the heat transfer results can be cleverly manipulated in the following form.

$$q_0 = \theta_0 W_1 \tanh(M_1) \quad \text{for one section} \tag{5}$$

$$q_0 = \theta_0 W_1 \frac{\tanh(M_1) + \frac{W_2}{W_1}\tanh(M_2)}{1 + \tanh(M_1)\frac{W_2}{W_1}\tanh(M_2)} \quad \text{for two sections} \tag{19}$$

$$q_0 = \theta_0 W_1 \frac{\tanh(M_1) + \frac{W_2}{W_1}\left[\frac{\tanh(M_2) + \frac{W_3}{W_2}\tanh(M_3)}{1 + \tanh(M_2)\frac{W_3}{W_2}\tanh(M_3)}\right]}{1 + \tanh(M_1)\frac{W_2}{W_1}\left[\frac{\tanh(M_2) + \frac{W_3}{W_2}\tanh(M_3)}{1 + \tanh(M_2)\frac{W_3}{W_2}\tanh(M_3)}\right]}$$

for three sections (25)

$$q_0 = \theta_0 W_1 \times \frac{\tanh(M_1) + \frac{W_2}{W_1}\left\{\frac{\tanh(M_2) + \frac{W_3}{W_2}\left[\frac{\tanh(M_3) + \frac{W_4}{W_3}\tanh(M_4)}{1 + \tanh(M_3)\frac{W_4}{W_3}\tanh(M_4)}\right]}{1 + \tanh(M_2)\frac{W_3}{W_2}\left[\frac{\tanh(M_3) + \frac{W_4}{W_3}\tanh(M_4)}{1 + \tanh(M_3)\frac{W_4}{W_3}\tanh(M_4)}\right]}\right\}}{1 + \tanh(M_1)\frac{W_2}{W_1}\left\{\frac{\tanh(M_2) + \frac{W_3}{W_2}\left[\frac{\tanh(M_3) + \frac{W_4}{W_3}\tanh(M_4)}{1 + \tanh(M_3)\frac{W_4}{W_3}\tanh(M_4)}\right]}{1 + \tanh(M_2)\frac{W_3}{W_2}\left[\frac{\tanh(M_3) + \frac{W_4}{W_3}\tanh(M_4)}{1 + \tanh(M_3)\frac{W_4}{W_3}\tanh(M_4)}\right]}\right\}}$$

for four sections (26)

Based on the above derivation, a numerical recursive formula can be observed to simulate a fin with more sections by just substituting $\frac{\tanh(M_i) + \frac{W_{i+1}}{W_i}\tanh(M_{i+1})}{1 + \tanh(M_i)\frac{W_{i+1}}{W_i}\tanh(M_{i+1})}$ into $\tanh(M_i)$.

Although all the derived results are based on the constant properties, all the results presented in this paper have been arranged into a similar formulation and hence a recursive formula can be observed and used to expand a fin into more sections. Finally, the results can be solved through numerical approach. This is the most precious contribution of the present paper.

Based on the previous derivation, the distributions of temperature and heat transfer rate at each section can be rearranged and written as the form

$$\theta_i = \theta_{i-1}\cosh(M_i) - \frac{q_{i-1}\sinh(M_i)}{W_i}, \quad \text{where } i = 1, n \tag{27}$$

$$q_i = q_{i-1} - (\theta_{i-1} + \theta_i) W_i \tanh\left(\frac{M_i}{2}\right), \quad \text{where } i = 1, n \tag{28}$$

It is noted that the heat transfer rate on the fin tip, q_n, obtained from Eq. (28) will approach zero. In addition to Eq. (27), the temperature on fin tip, θ_n, can also be solved directly from the following

$$\theta_n = \theta_{n-1} \cdot \frac{1}{\cosh(M_n)} \tag{29}$$

2.2. Convection boundary condition on fin tip

(1) One uniform section where $0 \leqq x \leqq l_1$.

If the heat transfer from fin tip is also considered, the heat transfer rate on the fin base is obtained as

$$q_0 = \theta_0 W_1 \left[\frac{\tanh(M_1) + N_1}{1 + \tanh(M_1) N_1}\right] \tag{30}$$

where W_1, M_1 are the same as before, and $N_1 = \frac{h_e}{m_1 k_1}$.

(2) Two uniform sections where $0 \leqq x \leqq (l_1 + l_2)$.

Following the same procedures, the heat dissipation of the fin containing two uniform sections is determined as

$$q_0 = \theta_0 W_1 \frac{\tanh(M_1) + \frac{W_2}{W_1}\left[\frac{\tanh(M_2) + N_2}{1 + \tanh(M_2) N_2}\right]}{1 + \tanh(M_1)\frac{W_2}{W_1}\left[\frac{\tanh(M_2) + N_2}{1 + \tanh(M_2) N_2}\right]} \tag{31}$$

(3) Three uniform sections where $0 \leqq x \leqq (l_1 + l_2 + l_3)$.

Similarly, the total heat dissipation for a fin including three uniform sections is determined as

$$q_0 = \theta_0 W_1 \times \frac{\tanh(M_1) + \frac{W_2}{W_1}\left\{\frac{\tanh(M_2) + \frac{W_3}{W_2}\left[\frac{\tanh(M_3) + N_3}{1 + \tanh(M_3) N_3}\right]}{1 + \tanh(M_2)\frac{W_3}{W_2}\left[\frac{\tanh(M_3) + N_3}{1 + \tanh(M_3) N_3}\right]}\right\}}{1 + \tanh(M_1)\frac{W_2}{W_1}\left\{\frac{\tanh(M_2) + \frac{W_3}{W_2}\left[\frac{\tanh(M_3) + N_3}{1 + \tanh(M_3) N_3}\right]}{1 + \tanh(M_2)\frac{W_3}{W_2}\left[\frac{\tanh(M_3) + N_3}{1 + \tanh(M_3) N_3}\right]}\right\}} \tag{32}$$

(4) N sections where $0 \leqq x \leqq (l_1 + l_2 + l_3 + \cdots + l_n)$.

Based on Eqs. (30)–(32), it can be observed that N_i appeared in i sections will be extended to $\frac{W_{i+1}}{W_i}\left[\frac{\tanh(M_{i+1}) + N_{i+1}}{1 + \tanh(M_{i+1}) N_{i+1}}\right]$ if the more $i+1$ sections are needed. The heat transfer rate of the whole fin, q_0, can also be solved with a known value of θ_0 or vice versa. For plotting the distributions of temperature and heat transfer rate, the individual term of θ_i and q_i can be written by following the same procedures as

$$\theta_i = \theta_{i-1}\cosh(M_i) - \frac{q_{i-1}\sinh(M_i)}{W_i}, \quad \text{where } i = 1, n \tag{27}$$

$$q_i = q_{i-1} - (\theta_{i-1} + \theta_i) W_i \tanh\left(\frac{M_i}{2}\right), \quad \text{where } i = 1, n \tag{28}$$

It is noted that the temperature on fin tip, θ_n, can also be solved directly from the following

$$\theta_n = \theta_{n-1} \cdot \frac{1}{\cosh(M_n) + N_n \sinh(M_n)} \tag{33}$$

3. Solution procedures

In solving the fin problem with variable thermal properties, the fin is firstly separated into many sections where each section has its constant properties and secondly the heat transfer results of the fin can be solved by a numerical algorithm. A numerical recursive formula has been obtained during the derivation of the solution of the fin problem with many sections where each section has its constant properties.

Using this model, a more accurate solution of a fin problem with variable thermal properties can be investigated by simply separating the fin into more sections. The numerical approach is executed in reverse direction by recursive formula. For the case with insulated fin tip, determine the value of $\dfrac{\tanh(M_i)+\frac{W_{i+1}}{W_i}\tanh(M_{i+1})}{1+\tanh(M_i)\frac{W_{i+1}}{W_i}\tanh(M_{i+1})}$ from the last term and then substitute it into $\tanh(M_i)$. As to the case with convective boundary condition on fin tip, determine the value of $\dfrac{W_{i+1}}{W_i}\left[\dfrac{\tanh(M_{i+1})+N_{i+1}}{1+\tanh(M_{i+1})N_{i+1}}\right]$ from the last term and substitute it into N_i. Continue these steps following in backward substitution until the simplest form like Eq. (5) or Eq. (30) is obtained. Finally, the heat transfer rate of the whole fin, q_0, can be solved with a known value of θ_0 or vice versa.

In solving the present problem, parameters W_i, M_i, and N_i must be calculated in advance. For this discrete model, the variable heat transfer coefficient h can be divided into some local constant heat transfer coefficients h_1, h_2, h_3, etc. to approximate the function of h. In the iterative calculation, the obtained temperature distribution, $\theta_0, \theta_1, \theta_2, \theta_3, \ldots, \theta_e$, will be substituted into the function of h to generate local heat transfer coefficients for the next iteration. And calculations will be stopped as the relative error of fin tip temperature within the last two iterations is less than 10^{-4}. This approach may also be applied with variable thermal conductivity k for fin by introducing $k_1, k_2, k_3, k_4, \ldots$ into M_i, and W_i.

4. Results and discussion

In order to perform the advantage of this mathematical model, an example is introduced to simulate a longitudinal fin that is under free convection and forced convection, separately. For the convection mechanism, variable heat transfer coefficient can be expressed in the form

$$h = h_{\mathrm{av.}}\frac{\theta^m}{l_c^n} \tag{34}$$

where θ is fin surface temperature in excess of environment. The values of $h_{\mathrm{av.}}$, m and n depend on properties of the liquid and flow mechanism [8,12]. Under free convection in laminar flow, values of m and n are all equal to 0.25, and $h_{\mathrm{av.}}$ is 1.42. Under free convection in turbulent flow, $m = 0.333$, $n = 0$, and $h_{\mathrm{av.}} = 1.31$. As to forced convection, the heat transfer coefficient is independent of the surface temperature but varied only by l_c. Therefore, $m = 0.$ and $n = 0.5$ for laminar flow and $n = 0.2$ for turbulent flow and typical value of $h_{\mathrm{av.}}$ is set between 25 W/m^2 K to 250 W/m^2 K. The data described above are listed in Table 2. Besides, m and n are set to 0 for the ideal case of a constant heat transfer coefficient.

Furthermore, here defines a parameter of thermal resistance R which is expressed as

$$R = \frac{\theta_0}{q_0} \tag{35}$$

Table 2
Constant for simplified equation from vertical surface to air at atmospheric pressure, according to [8,12]

Type of flow	$h_{\mathrm{av.}}$, m, n	
	Free convection	Forced convection
Laminar flow	1.42, 0.25, 0.25	$25 \leqq h_{\mathrm{av.}} \leqq 250$, 0, 0.5
Turbulence flow	1.31, 0.333, 0	$25 \leqq h_{\mathrm{av.}} \leqq 250$, 0, 0.2

Table 3
Comparison between the present study and Liaw and Yeh [9]

Distance, x Temperature distribution, θ_n Values of m and cases		0.0	0.2	0.4	0.6	0.8	1.0	Range of error, %
$m = -0.25$	The present study	0.624	0.638	0.681	0.754	0.859	1.000	0.000
	[9]	0.624	0.638	0.681	0.754	0.859	1.000	
$m = 0.25$	The present study	0.668	0.680	0.717	0.780	0.873	1.000	0.001
	[9]	0.667	0.679	0.716	0.780	0.873	1.000	
$m = 2.0$	The present study	0.752	0.760	0.786	0.832	0.901	1.000	0.001
	[9]	0.751	0.760	0.786	0.832	0.901	1.000	
$m = 3.0$	The present study	0.752	0.760	0.786	0.832	0.901	1.000	0.001
	[9]	0.751	0.760	0.786	0.832	0.901	1.000	

The value of R is also considered as the performance of fin, the less value of R indicates the better performance of fin.

For validating the present approach, the study treated from Liaw and Yeh [9] is introduced here for comparison. From Table 3, both results are quite similar where the maximum difference is only 0.001 with $m = -0.25$, 0.25, 2.0, and 3.0. In addition, Ma et al. [7] used Fourier series approach to investigate a rectangular fin with arbitrary heat transfer coefficient on fin surface. For comparison with the present study by setting the same fin geometry and convective condition, the results shown in Table 4 by thermal transmission ratio, μ, defined in Ref. [7] which is the reciprocal of thermal resistance are identical.

Table 4
Comparison between the present study and Ma et al. [7]

	The present study	Ma et al. [7]	Error
Thermal transmission ratio, μ	0.1150	0.1150	0.0%

Fin normal cross-section area, a, is $10\ \text{mm} \times 10\ \text{mm}$ and fin thermal conductivity k is taken to be 100 W/m K, heat transfer coefficient h is assumed to be 50 W/m^2 K along the first 10 mm of its length, and 100 W/m^2 K along the remaining 30 mm.

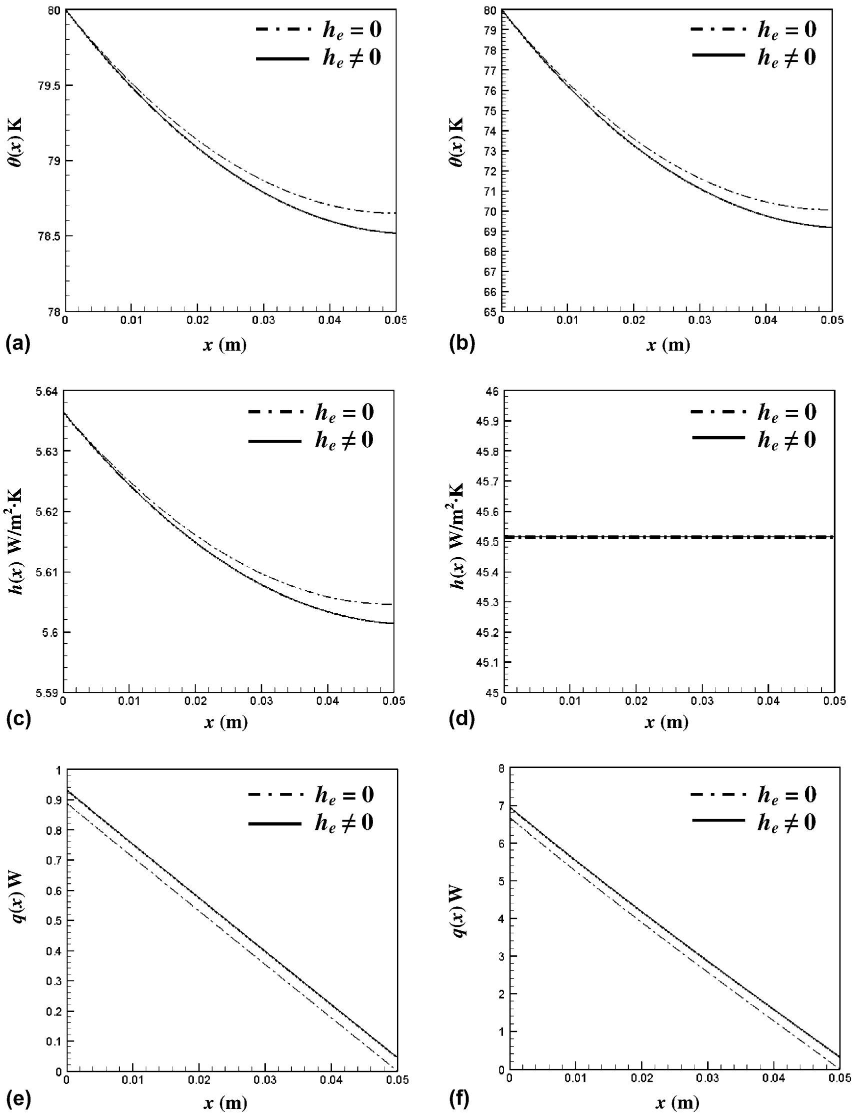

Fig. 3. Distribution of θ, h, and q on location x with input constant temperature difference $\theta_0 = 80$ K for fin base, where fin length, $l = 0.05$ m, fin thickness is 0.01 m, fin width is 0.01 m and material is Duralumin, $k = 164$ W/m K; (a), (c), (e) for free convection, $h_{av.} = 1.31$ W/m^2 K, and $m = 0.333$, $n = 0$; (b), (d), (f) for forced convection, $h_{av.} = 25$, and $m = 0$, $n = 0.2$.

For an example of fin, the temperature difference between fin base and the surrounding is assumed as 80 K. Based on the discrete model, fin length is divided into 1000 sections for calculation. Temperature distribution, heat transfer coefficients, and heat transfer rate versus relative location are graphically shown in Fig. 3. The effect of fin tip with or without convective condition is also taken into account and shown in Fig. 3. It can be observed from Fig. 3(a) and (b) that the lower temperature difference on fin tip is happened to forced convection. Based on Fig. 3(d), heat transfer coefficient along fin length is a constant for forced convection. And relative to free convection, heat transfer rate dissipated by fin under forced convection is greater shown in Fig. 3(e) and (f).

The thermal resistance, R, for any fin length, l, can be obtained as if the parameters of fin volume, V, and fin width, γ, are given. Fig. 4 shows thermal resistance, R, versus fin length, l, for variable heat transfer coefficients along fin surface. Take an observation of Fig. 4(a) for free convection mode, deviation of thermal resistance between copper and Duralumin is pronounced when fin length is becoming longer. Meanwhile, the minimum value of thermal resistance can be found from Fig. 4(a). The same results are also occurred to forced convection mode shown in Fig. 4(b).

Fig. 5 shows temperature distributions calculated by the present study which are also appeared in Kern and Kraus [1]. The variable heat transfer coefficient is taken as

$$h = (\beta + 1)h_{\text{av.}}\left(\frac{x}{l}\right)^{\beta} \tag{36}$$

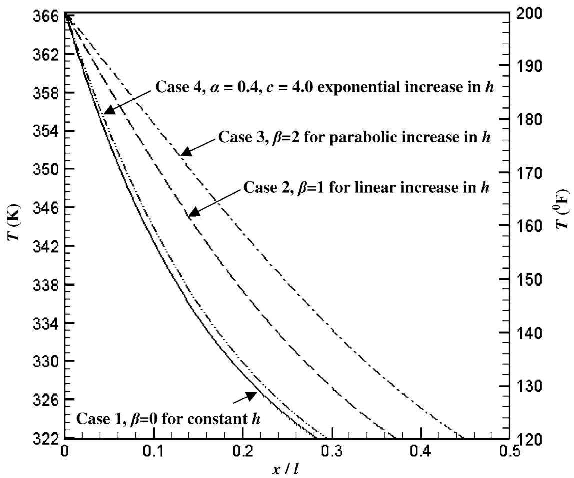

Fig. 5. Temperature distribution of longitudinal rectangular fin for variable heat transfer coefficients, where $l = 0.1016$ m (4 in.) long, $\gamma = 0.3048$ m (1 ft) wide, $\delta = 0.003175$ m (0.125 in.) thick, $k = 34.61$ W/m K (20 Btu/ft h °F), and $h_{\text{av.}} = 170.34$ W/m^2 K (30 Btu/ft^2 h °F), ambient temperature is 310.78 K (100 °F), $h_e \neq 0$.

for cases 1–3 where $\beta = 0$ for constant h, $\beta = 1$ for linear increase in h, and $\beta = 2$ for parabolic increase in h. The variable heat transfer coefficient is also taken as

$$h = h_{\text{av.}}\left[\frac{1 - \alpha e^{-c(x/l)}}{1 - (\alpha/c)(1 - e^{-c})}\right] \tag{37}$$

for case 4 where α and c are dimensionless parameters for exponential increase in h. Based on Fig. 5, temperature distributions for cases 2–4 are higher than that for case 1. This result could be expected because that heat

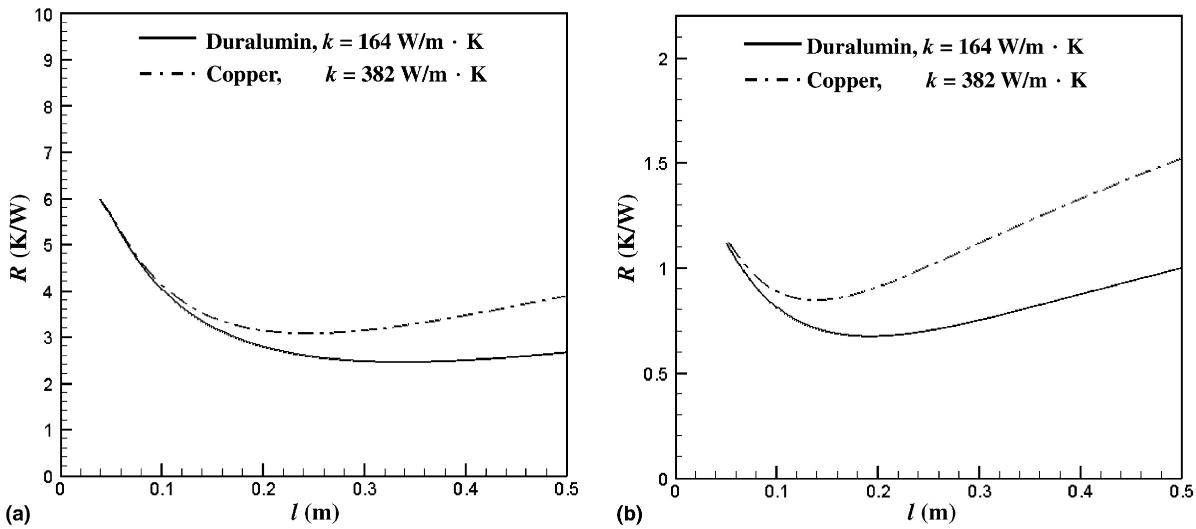

Fig. 4. Dependence of thermal resistance, R, versus fin length, l, for the free convection and forced convection and $V = 0.00015$ m^3, $\gamma = 0.15$ m, $\theta_0 = 80$ K, $h_e \neq 0$: (a) free convection, $h_{\text{av.}} = 1.42$, $m = 0.25$, $n = 0.25$; (b) forced convection, $h_{\text{av.}} = 25$, $m = 0$, $n = 0.2$.

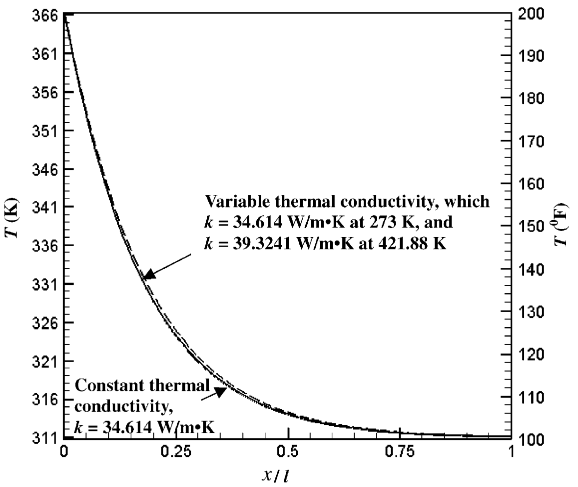

Fig. 6. Temperature distribution of longitudinal rectangular fin for variable thermal conductivity, where $l = 0.1016$ m (4 in.) long, $\gamma = 0.3048$ m (1 ft) wide, $\delta = 0.003175$ m (0.125 in.) thick, $h_{av.} = 170.34$ W/m^2 K (30 Btu/ft^2 h °F), and fin base temperature is 366.33 K (200 °F), ambient temperature is 310.78 K (100 °F), $h_e \neq 0$.

transfer coefficients near the fin base region in these cases are lower than that in case 1.

In this study, thermal conductivity, k, can be attempted to be a variable property. Refer to Kern and Kraus [1], thermal conductivity, k, is assumed to be

$$k = k_0\left(1 + \varepsilon\frac{\theta}{\theta_0}\right) \tag{38}$$

where k_0 is thermal conductivity at T_0, and ε is the slope of temperature-dependent thermal conductivity curve. The effect of variable thermal conductivity is shown in Fig. 6. The temperature distribution of variable thermal conductivity is slightly higher than that of constant thermal conductivity because thermal conductivity of the former is always greater than that of the latter. It is worth to mention here that the results shown in Figs. 5 and 6 are almost the same as Kern and Kraus [1] of pages 339 and 343.

Although copper fin has good thermal dissipation performance, the high density and cost of copper are its disadvantage. From Fig. 7, a longitudinal rectangular fin is composed of two materials, copper and Duralumin, for two arrangements. For the fixed profile and volume of fin with $V = 0.000037$ m^3, $\gamma = 0.15$ m, $l = 0.2$ m, the results are expressed in Fig. 8 where the thermal resistance, R, is observed with the proportion of volume for Duralumin. Although proportion of volume for Duralumin is the same, arrangement A can achieve better performance for the cases of free convection and forced convection.

In 1987, Ünal [13] quoted the previous studies and applied an analytical expression for the one-dimensional temperature distribution in a pin fin or a straight fin with rectangular profile due to boiling heat transfer. Fin is made of copper with thermal conductivity of 382 w/m K, and outer diameter of 0.635 cm. The heat transfer coefficient is set as

$$h = \rho_j\theta^{\sigma_j} \tag{39}$$

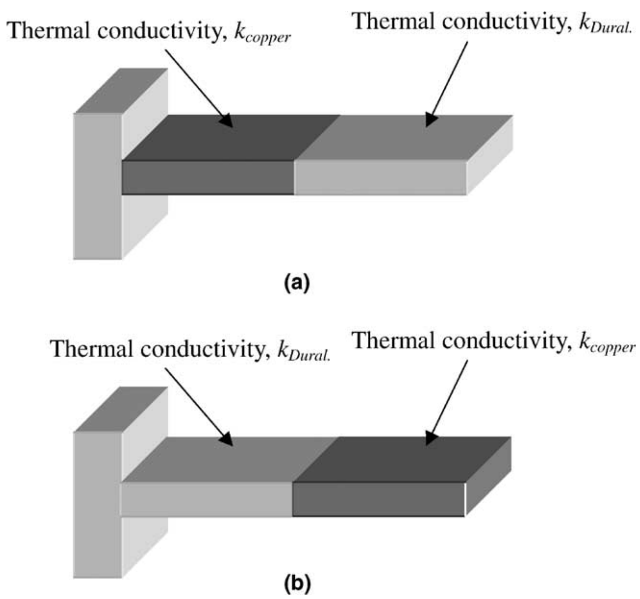

Fig. 7. The arrangements of materials for longitudinal fin: (a) arrangement A (copper–duralumin), (b) arrangement B (duralumin–copper).

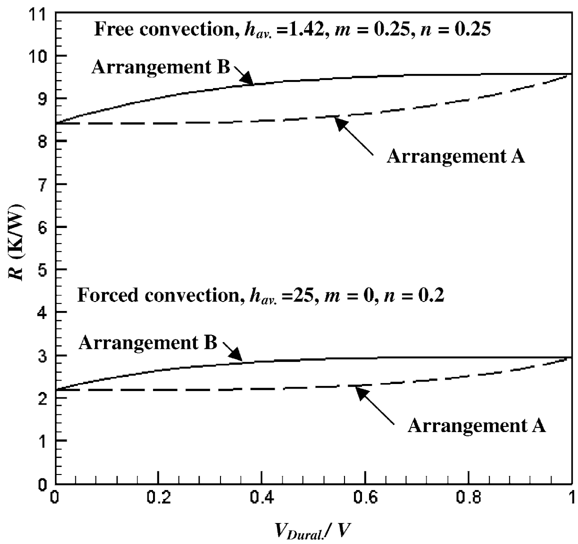

Fig. 8. Thermal resistance, R, of compound materials where the fin parameters are $V = 0.000037$ m^3, $\gamma = 0.05$ m, $l = 0.2$ m, $h_e \neq 0$, $\theta_0 = 80$ K, and ambient temperature is 293 K. Thermal conductivity of Duralumin is $k = 164$ W/m K at 293 K and $k = 182$ W/m K at 373 K. Thermal conductivity of copper is $k = 382$ W/m K at 293 K and $k = 379$ W/m K at 373 K.

Table 5
Reference data from Ünal [13] of Eq. (39) for various types of boiling

Type of boiling	ρ_j, σ_j	
	R113	Isopropyl alcohol
Nucleate boiling, $\theta < \theta_d$, $j = 1$	16.0, 2.0	28.0, 2.0
Transition boiling, $\theta_d \leqq \theta < \theta_f$, $j = 2$	3.0×10^9, -4.0	4.7×10^7, -2.5
Film boiling, $\theta_f \leqq \theta$, $j = 3$	194.0, 0	254.0, 0
θ_d, θ_f (K)	22.0, 71.0	22.7, 81.0

where ρ_j, and σ_j, are related to θ_f, and θ_d given in Table 5, and θ is temperature difference between fin surface and saturated temperature. In Ünal [13], a quite reasonable agreement is found between the analytical results from theory and the experimental data. The application of various types of boiling mode which occurs simultaneously at adjacent positions on pin fin surface can also be done by the present study using discrete model. Relationships between fin base temperature in excess of saturated temperature, θ_b, and base temperature gradient, $-q_0/ka$, are depicted in Fig. 9. This calculation was carried out through 200 sections of discrete fin model. The curve trend and boiling process are identical to those from Ünal [13].

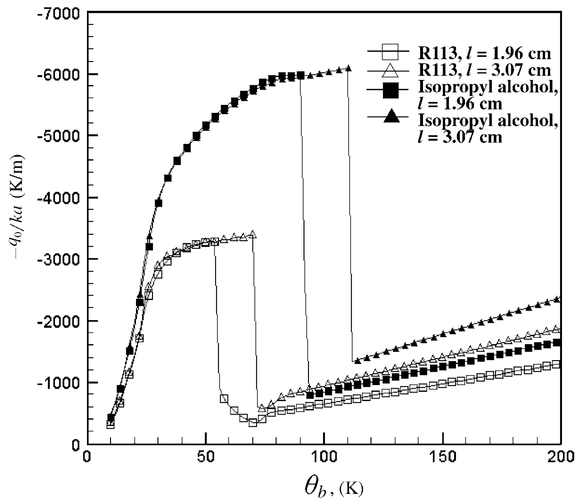

Fig. 9. Dependence of temperature gradient, $-q_0/ka$, on slowly raised base temperature difference, θ_b, for fin which is immersed in R113 and Isopropyl alcohol at atmospheric pressure.

5. Conclusions

This paper presents a new approach to calculate temperature distribution and heat transfer rate of a singular fin where thermal properties such as heat transfer coefficient, thermal conductivity can be considered as the position-dependent or temperature-dependent function. By using discrete model in the present approach, the recursive formulas can be obtained for both conditions with and without heat transfer on fin tip that can be easily implemented into real applications. Meanwhile, several examples including composite of a fin have been successfully simulated to demonstrate the validity of the present approach. Finally, example including different types of boiling modes which occur simultaneously at adjacent positions on a pin fin surface is explored to demonstrate the distribution of its temperature gradient on the fin surface.

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